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The Wave Group on Asymptotically Hyperbolic Manifolds

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We show that the wave group on asymptotically hyperbolic manifolds belongs to an appropriate class of Fourier integral operators. Then we use now standard techniques to analyze its (regularized) trace. We prove that, as in the case of compact manifolds without boundary, the singularities of the regularized wave trace are contained in the set of periods of closed geodesics. We also obtain an asymptotic expansion for the trace at zero. © 2001 Academic Press

1. INTRODUCTION

The spectral and scattering theory of asymptotically hyperbolic manifolds have been extensively studied in recent years. See for example [1, 3, 10, 11, 16, 20, 21, 25] and references cited there. They are a very good example of a class of manifolds for which a geometric scattering theory can be developed. They have also been studied in connection with conformal field theory; see for example [2, 9, 28] and references cited there. In this note, we study the wave group on asymptotically hyperbolic manifolds and show that it belongs to an appropriate class of Fourier integral operators. As an application, we analyze the singularities of its (regularized) trace. As in the

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case of scattering on the Euclidean space, it is expected that this group and its trace will give information about the manifold.

A compact manifold, X, with boundary ∂X, is called asymptotically hyperbolic when it is equipped with a metric of the form

$$g = \frac{H}{x^2},$$

where x is a defining function of ∂X and H is a smooth Riemannian metric on X, nondegenerate up to ∂X, and such that $|dx|_H = 1$ at ∂X. This name is due to the fact, see [21], that along a smooth curve in $X \backslash \partial X$, approaching a point at ∂X, all sectional curvatures of g approach -1. The simplest examples of such manifolds are the hyperbolic space, $\mathbb{H}^{n+1}$, and its quotients by certain group actions; see for example Section 8 of [21].

It is proved in Proposition 2.1 of [16] that, under these assumptions on g, there exists a product decomposition $X \sim \partial X \times [0, \varepsilon)$, for ε small enough, such that

$$g = \frac{dx^2 + h(x, y, dy)}{x^2}. \tag{1.1}$$

Let Δ denote the (positive, self-adjoint) Laplacian corresponding to the asymptotically hyperbolic metric g, acting on half-densities. We recall that g induces a canonical trivialization of the 1-density bundle by taking $\theta = \sqrt{\operatorname{vol}(g)}\, |dx\, dy|$, the Riemannian density. The square root of this is then a natural trivialization of the half-density bundle. The Laplacian is defined by

$$\Delta(f\theta^{1/2}) = (\Delta f)\, \theta^{1/2},$$

where the Laplacian on the right hand side is the usual one acting on functions. It is well known, see for example [19, 21], that the continuous spectrum of Δ is $[n^2/4, \infty)$.

The sections of the density bundle ${}^0\Omega(X)$ are defined to be smooth multiples of the Riemannian half-density. In local coordinates where (1.1) holds it is given by

$$\theta = f(x, y)\, \frac{dx}{x} \frac{dy}{x^n}, \qquad f \in C^\infty(X), \qquad f \neq 0.$$

The bundle ${}^0\Omega^{1/2}(X)$ is the half-density bundle obtained from ${}^0\Omega(X)$. Similarly we define the bundle ${}^0\Omega^{1/2}(X \times X)$.

The group $\cos(t\sqrt{\Delta - n^2/4})$ is defined to be the operator whose kernel $U(t, w, w')$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} + \Delta_w - \frac{n^2}{4}\right) U(t, w, w') = 0,$$
$$U(0, w, w') = \delta(w, w'), \qquad \frac{\partial}{\partial t} U(0, w, w') = 0. \tag{1.2}$$

Here $\delta(w, w')$ acts on half densities $f\theta'^{1/2}$ according to

$$f(w)\,\theta^{1/2} = \int \delta(w, w')\, f(w')\, \theta^{1/2}(\omega)\, \theta'^{1/2}(\omega')$$

As already mentioned, our goal is to study the structure of $\cos(t\sqrt{\Delta - n^2/4})$. Once we show that it belongs to a class of Fourier integral operators, the arguments of [7, 13, 14] can be applied to analyze the singularities of its (regularized) trace. This is related to the behaviour of the scattering phase at high energies and the possible existence of a Poisson type formula relating the wave group and the resonances in this setting. See for example [10, 11] for the case of Riemann surfaces. The wave group for hyperbolic space has been studied in [12, 17, 18]. It is possible that the analysis of the trace may be carried out using techniques as in [6], where asymptotically Euclidean manifolds were considered.

We are heavily influenced by Melrose's work in the b-category [22], though he does not examine the specific problem of constructing wave groups there, and by his work with Mazzeo [21] on the construction of the resolvent for this class of manifolds.

In [21] it was shown that the resolvent for the Laplacian can be constructed within a "large" calculus of zero pseudo-differential operators. In particular the Schwartz kernels of these operators were constructed as distributions on the blown-up space $X \times_0 X$, obtained by taking $X \times X$ and blowing up $D_{\partial X}$, which is the intersection of the diagonal D with the corner $\partial X \times \partial X$.

We recall that blow-up is really just an invariant way of introducing polar coordinates and that a function is smooth on the space $X \times_0 X$ if it is smooth in polar coordinates about $D_{\partial X}$. As a set, $X \times_0 X$ is $X \times X$ with $D_{\partial X}$ replaced by the interior pointing portion of its normal bundle. Let

$$\beta\colon X \times_0 X \to X \times X$$

denote the blow-down map. If (x, y) are coordinates in a product decomposition of X near ∂X, and we let (x', y') be the corresponding coordinates

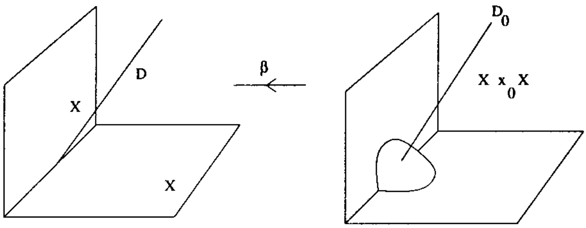

FIG. 1. $X \times_0 X$.

on a second copy of X, then $R = (x^2 + x'^2 + (y - y')^2)^{1/2}$ is a defining function for a new face, which we call the front face, $\mathscr{F}$. This is the lift of $D_{\partial X} = D \cap (\partial X \times \partial X)$. The functions $\rho = x/R$ and $\rho' = x'/R$ are then defining functions for the other two boundary faces which we call the top face $\mathscr{T}$ and bottom face $\mathscr{B}$, respectively. See Fig. 1, which is taken from Section 3 of [21]. One advantage of working on this blown-up space is that the lift of the diagonal of $X \times X$ only meets the boundary of the blown-up space at $\mathscr{F}$ and is disjoint from the other two boundary faces.

We define the bundle ${}^0\Omega^{1/2}(X \times_0 X)$ to be the lift of ${}^0\Omega^{1/2}(X \times X)$ under the blow-down map β.

The kernel K of a zero pseudo-differential operator in the class $\Psi_0^{m, a, b}(X, {}^0\Omega^{1/2}(X))$, $m \in \mathbb{R}$, $a, b \in \mathbb{C}$, is defined in [21] to be a distribution which can be written as $K = K_1 + K_2$, where the lift of K_1 to $X \times_0 X$ is conormal of order m to the lifted diagonal, D_0, and smooth up to the front face, and vanishes to infinite order at the top and bottom faces. The second part, K_2, is of the form $K_2 = \rho^a \rho'^b F$, $F \in C^\infty(X \times_0 X, {}^0\Omega^{1/2}(X \times_0 X))$, where ρ and ρ' are defining functions of the top and bottom faces respectively.

Based on this construction, it is natural to look for the wave group to have a Schwartz kernel which is nice on the space $\mathbb{R} \times (X \times_0 X)$.

We also need to recall the definition of the normal operator from [21]. Let $p \in \partial X$ and let X_p be the inward pointing vectors in the tangent space to X at p, $T_p(X)$. As observed in [21], this is a manifold with boundary and has a metric

$$g_p = (dx)^{-2} H_p, \tag{1.3}$$

where $g = x^{-2} H$, making it isometric to the hyperbolic upper half-plane. (We regard H_p and dx as linear functions on the tangent space X_p.) Mazzeo and Melrose observed that the leaf of the front face above a point p is naturally isomorphic to X_p, using a natural group action on the front face. This group action is obtained by lifting the action of the subgroup of the general linear group of the boundary of X_p to the normal bundle of X_p, as a leaf of the front face is just a quarter of the normal bundle over p.

Let F_p be the fibre of the front face lying over the point $(p, p) \in D \cap (\partial X \times \partial X)$. Since the kernel, $k(B)$, of an element $B \in \Psi_0^{m, a, b}(X)$, is conormal to the lift of the diagonal D_0, it can be restricted to F_p and the kernel of the normal operator, $N_p(B)$, is defined by

$$k(N_p(B)) = k(B)|_{F_p}. \tag{1.4}$$

Let (x, y) be local coordinates near $p \in \partial X$, with x a boundary defining function and also denote the natural corresponding linear coordinates on X_p by (x, y). Let (x', y') be the same coordinates on the right factor in $X \times X$ and let $s = x/x'$, $z = (y - y')/x$. Then if the Schwartz kernel of a map B is $k(x', y', s, z)\, \gamma$ with $\gamma = |ds\, dz\, dx\, dy/sx^{n+1}|^{1/2}$, the normal operator is given at $p = (0, \bar{y})$ by

$$[N_p(B)(f\mu)] = \int k(0, \bar{y}, s, z)\, f\left(\frac{x}{s}, y - \frac{x}{s} z\right) \frac{ds}{s}\, dz\,.\mu, \tag{1.5}$$

where $\mu = |dx/x\, dy/x^n|^{1/2}$.

As observed in [21], each fiber F_p of the front face $\mathscr{F}$ has a natural origin 0_p, which in coordinates s, z is given by $0_p = \{s = 1, z = 0\}$. For example, we find that the kernel of the identity is

$$K(\mathrm{Id}) = \delta(s-1)\, \delta(z)\, \gamma,$$

and its normal operator is

$$N_p(\mathrm{Id}) = \delta(s-1)\, \delta(z) \left| \frac{ds}{s}\, dz\, \frac{dx}{x} \frac{dy}{x^n} \right|^{1/2} = \delta(0_p) \left| \frac{ds}{s}\, dz\, \frac{dx}{x} \frac{dy}{x^n} \right|^{1/2}.$$

The fibre F_p itself is a manifold with corners which are defined by its intersection with the top and bottom faces. Proposition 5.19 of [21] gives that kernel of the normal operator is of the form $k(N_p(B)) = k_1(N_p(B)) + k_2(N_p(B))$ where $k_1(N_p(B))$ is conormal to the intersection of the lifted diagonal and F_p, and vanishes to infinite order at the boundaries, whilst $k_2(N_p(B)) = \rho^a_{F_p} \rho'^b_{F_p} F'$, $F \in C^\infty(F_p)$, $\rho_{F_p} = \rho\,|_{F_p}$, $\rho'_{F_p} = \rho'\,|_{F_p}$.

In section 3 we extend the notion of normal operators to Fourier integral operators.

The fundamental fact that enables the simplicity of this note is the finite speed of propagation of information which ensures that there is no support on the top and bottom faces and only on the interior of the front face.

The same techniques used here apply to $(\sqrt{\Delta - n^2/4})^{-1} \sin(t\sqrt{\Delta - n^2/4})$, $t \in \mathbb{R}$, by just switching the initial conditions, but it is not so clear how to proceed with $\sin(t\sqrt{\Delta - n^2/4})$, as it would seem to have support on all faces, as $\sqrt{\Delta - n^2/4}$ is not a local operator.

To construct the wave group we introduce the class of 0-Fourier integral operators. It is not our purpose to discuss the most general Fourier integral operators that could be defined in this setting. Instead we will only consider those whose kernels, when lifted to $X \times_0 X$, have support away from the top and bottom faces. So, we are able to ignore the corners formed by the intersections of the front face with the top and bottom faces. Hence, for this particular class of operators, we only need to consider Lagrangian submanifolds on a manifold with boundary. As mentioned above, these operators are closely related to the b-Fourier integral operators introduced in [22]. A somewhat simplified exposition of b-Fourier integral operators and their application to the study of propagation of singularities for semilinear wave equations can be found in [27].

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2. THE 0-STRUCTURE

As observed in [21], the Laplacian on an asymptotically hyperbolic manifold is a second order operator which locally is the product of vector fields that vanish at ∂X. Here, in analogy with [22], where the b-Fourier integral operators were introduced to study properties of differential operators defined by products of vector fields tangent to ∂X, we will define the 0-Fourier integral operators. First we need to recall the definition of the 0-cotangent bundle on X and its symplectic structure. This follows [21, 22, 23, 24, 26].

On a C^∞ manifold with boundary X the space $\mathcal{V}_0(X)$ of smooth vector fields that vanish on the boundary is a Lie algebra. If we take local coordinates $(x, y_1, ..., y_n)$, in which x is a defining function of ∂X, $\mathcal{V}_0(X)$ it has the local basis $x\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y_j}$, $1 \leqslant j \leqslant n$, near ∂X, and so it is the space of all C^∞ sections of a vector bundle over X:

$$\mathcal{V}_0(X) = C^\infty(X, {}^0TX).$$

Restriction to the interior extends to define a smooth bundle map

$$\iota\colon {}^0TX \to TX,$$

which is an isomorphism in the interior and vanishes over ∂X. Let ${}^0T^*X$ be the dual bundle to ${}^0TX$. The map ι then induces a map ι^*, which in dual coordinates (x, ξ, y, η) is given by

$$\iota^*\colon T^*X \to {}^0T^*X$$
$$(x, \xi, y, \eta) \mapsto (x, x\xi, y, x\eta) = (x, \lambda, y, \mu).$$

The canonical 1-form in T^*X,

$$\alpha = \xi \, dx + \eta \cdot dy,$$

is pulled back to

$$(2.1) \qquad {}^0\alpha = \frac{\lambda}{x} \, dx + \frac{\mu}{x} \cdot dy \in C^\infty({}^0T^*X; {}^0T^*({}^0T^*X)).$$

This leads us to

DEFINITION 2.1. A 0-canonical transformation between C^∞ manifolds with boundary X and Y is a C^∞ homogeneous map $\chi\colon \Gamma \subset {}^0T^*X \to {}^0T^*Y$ defined on an open conic set $\Gamma \subset {}^0T^*X \backslash 0$, such that $\chi^{*0}\alpha_Y = {}^0\alpha_X$.

It is clear that a 0-canonical transformation maps ${}^0T^*_{\partial X}X$ to ${}^0T^*_{\partial Y}Y$.

As in [22] we want to show that certain 0-canonical transformations from X to X define Lagrangian submanifolds of $T^*(X \times_0 X)$. The methods used here are a natural modification of the ones from [22], see also [27].

Recall that $\mathscr{F}$ denotes the front face of $X \times_0 X$. Let $(X \times_0 X)_{\mathrm{d}}$ denote the doubling of $X \times_0 X$ across $\mathscr{F}$, and let $T^*_{\mathscr{F}}(X \times_0 X)$ denote the restriction of the cotangent space to the front face. We say that a smooth conic closed Lagrangian submanifold $\Lambda \subset T^*(X \times_0 X)$ is extendible if it intersects $T^*_{\mathscr{F}}(X \times_0 X)$ transversally. In that case there exists a smooth conic closed Lagrangian submanifold $\Lambda_e \subset T^*(X \times_0 X)_{\mathrm{d}}$ such that

$$(2.2) \qquad \Lambda = \Lambda_e \cap T^*(X \times_0 X), \qquad \Lambda_0 = \Lambda \pitchfork T^*_{\mathscr{F}}(X \times_0 X).$$

To consider the implications of the transversality in (2.2), let $(x, y_1, ..., y_n)$ denote local coordinates in $X \times_0 X$, which are valid near a point on the front face and in which x is a defining function of this boundary. Let ξ and η_j denote the respective dual variables. Thus (x, y, ξ, η) give local coordinates in $T^*(X \times_0 X)$, near the front face, with x a defining function of the boundary. Condition (2.2) means that dx must be nonvanishing on Λ. So x and some selection of (y, ξ, η) must give local coordinates on Λ. The canonical two form $\omega = dx \wedge d\xi + \sum_j dy_j \wedge d\eta_j$ must vanish on Λ up to Λ_0. Hence $d\xi$ must be a multiple of dx when acting on $T_{\Lambda_0}\Lambda$. Thus there must exist a function $\phi(x, y, \eta)$ such that

$$\Lambda \subset \{\xi = x\phi(x, y, \eta)\}.$$

Thus it follows that $\xi = 0$ on Λ_0 and $\sum_j dy_j \wedge d\eta_j = 0$ on $T\Lambda_0$. Thus we have shown:

LEMMA 2.1. *Let $\Lambda \subset T^*(X \times_0 X)$ be an extendible Lagrangian. Then $\Lambda_0 = \Lambda \cap T^*_{\mathscr{F}}(X \times_0 X)$ is a Lagrangian submanifold of $T^*\mathscr{F}$.*

We want to characterize the canonical transformations χ: ${}^0T^*X \to {}^0T^*X$ that induce extendible Lagrangians on $T^*(X \times_0 X)$. We recall, see for example chapter 21 of [15], that the graph of a canonical transformation is a Lagrangian submanifold of ${}^0T^*X \times {}^0T^*X$ with the symplectic form

$$\omega = \pi_1^* \omega_X - \pi_2^* \omega_X, \tag{2.3}$$

where ω_X is the symplectic form on ${}^0T^*X$ and π_j: ${}^0T^*X \times {}^0T^*X \to {}^0T^*X$, $j = 1, 2$, is the projection on the jth copy of ${}^0T^*X$. Henceforth, we will refer to ω, as defined in (2.3), as the symplectic structure on ${}^0T^*X \times {}^0T^*X$.

Next we observe that the dual to the blow-down map β: $X \times_0 X \to X \times X$ induces a smooth map

$$T^*X \times T^*X \sim T^*(X \times X) \to T^*(X \times_0 X) \tag{2.4}$$

which is an isomorphism in the interior.

PROPOSITION 2.1. *Let χ: ${}^0T^*X \to {}^0T^*X$, be a homogeneous canonical transformation whose projection onto the base space is the identity when restricted to ∂X. The identification $T^*X \times T^*X \sim {}^0T^*X \times {}^0T^*X$, with the symplectic structure given by* (2.3), *over the interior combined with* (2.4) *gives a C^∞ map*

$$ {}^0T^*X \times {}^0T^*X \to T^*(X \times_0 X) \qquad \textit{over} \quad \mathring{X} \times \mathring{X} \tag{2.5}$$

which restricted to the graph of χ extends by continuity and embeds it as a smooth Lagrangian submanifold of $T^*(X \times_0 X)$, denoted by Λ_χ. Moreover, Λ_χ intersects the boundary of $T^*(X \times_0 X)$ only over $T^*_{\mathscr{F}}(X \times_0 X)$, it is extendible across the front face, and

$$\Lambda_{\chi_0} = \Lambda_\chi \pitchfork T^*_{\mathscr{F}}(X \times_0 X)$$

is a Lagrangian submanifold of $T^*\mathscr{F}$.

Proof. To prove this we use local coordinates (x, y), (x', y') valid near ∂X. Let (x, y, ξ, η), (x', y', ξ', η'), (x, y, λ, μ), and (x', y', λ', μ') be the corresponding canonical dual coordinates in T^*X and ${}^0T^*X$ respectively.

Near the front face we can use projective coordinates given by

$$s = \frac{x}{x'}, \qquad z = \frac{y - y'}{x'}.$$

Under the map (2.5) the 1-forms

$$\frac{\lambda}{x}\,dx-\frac{\lambda'}{x'}\,dx'+\frac{\mu}{x}\cdot dy-\frac{\mu'}{x'}\cdot dy' \qquad \text{and} \qquad \sigma\,ds+\tau\,dx'+\zeta\cdot dz+\tilde{\eta}\cdot dy'$$

are identified. So,

$$\begin{aligned} \sigma&=\frac{\lambda x'}{x}, \qquad \tau=\frac{\lambda}{x'}-\frac{\lambda'}{x'}-\mu\cdot\frac{y-y'}{x'} \\ \zeta&=\frac{x'\mu}{x}, \qquad \tilde{\eta}=\frac{\mu}{x}-\frac{\mu'}{x'}. \end{aligned} \tag{2.6}$$

Since the projection of χ to the base space is the identity when restricted to ∂X, its graph is of the form

$$\begin{aligned} \operatorname{graph}\chi=\Big\{((x,y,\lambda,\mu);(x',y',\lambda',\mu'));\ x'=xa,\ y'=y+xY \\ \lambda'=\lambda+\frac{1}{a}\,\mu\cdot Y+x\Lambda,\ \mu'=a\mu+xH\Big\}, \end{aligned} \tag{2.7}$$

where $a>0$, Y, Λ, and H are C^∞ functions of (x, y, λ, μ). From this and (2.6) it follows that

$$\sigma=a\lambda, \qquad \tau=\frac{\Lambda}{a}, \qquad \zeta=a\mu, \qquad \tilde{\eta}=-\frac{H}{a}. \tag{2.8}$$

Since $a=\frac{x'}{x}$ is smooth and positive on Λ_χ, this shows that the map (2.5) extends smoothly to

$$\operatorname{graph}\chi\leftrightarrow\Lambda_\chi\hookrightarrow T^*(X\times_0 X) \tag{2.9}$$

and that Λ_χ only intersects the boundary of $T^*(X\times_0 X)$ over the front face $\mathscr{F}=\{x'=0\}$ and does so transversally. Therefore it is extendible across $\mathscr{F}$. It follows from Lemma 2.1 that $\Lambda_{\chi,0}$ is a Lagrangian submanifold of $T^*\mathscr{F}$. ∎

Definition 2.2. A homogeneous canonical transformation $\chi\colon {}^0T^*X\to{}^0T^*X$, whose projection onto the base space is the identity when restricted to ∂X, will be called a liftable canonical transformation.

We will apply this result to the Lagrangian submanifold defined by the flow of the Hamiltonian vector field of a function $p\in C^\infty({}^0T^*X)$.

For the canonical 1-form ${}^0\alpha$ given by (2.1), let ${}^0\omega = d{}^0\alpha$ be the canonical 2-form. Given $p \in C^\infty(T^*X)$ the 0-Hamiltonian vector field of $p, {}^0H_p$, is defined by

$$ {}^0\omega(\,\bullet\,, {}^0H_p) = dp. \tag{2.10}$$

In local coordinates where ${}^0\alpha$ is given by (2.1), ${}^0H_p$ is given by

$$ {}^0H_p = x\frac{\partial p}{\partial\lambda}\frac{\partial}{\partial x} + x\sum_{j=1}^{n}\frac{\partial p}{\partial\mu_j}\frac{\partial}{\partial y_j} - \left(x\frac{\partial p}{\partial x} + 2p\right)\frac{\partial}{\partial\lambda} - \sum_{j=1}^{n}\left(x\frac{\partial p}{\partial y_j} - \frac{\partial p}{\partial\lambda}\mu_j\right)\frac{\partial}{\partial\mu_j}. $$

Observe that the projection of ${}^0H_p$ to the base space X vanishes at ∂X.

Based on the usual theory of Fourier integral operators, the Lagrangian we are interested in is the one generated by the flow of the Hamiltonian vector field of the length function $g \in C^\infty({}^0T^*X)$ given by the metric g. Using 1. we write the metric

$$ g = \frac{dx^2 + H(y, dy) + xh_1(x, y, dy)}{x^2}, \qquad x > 0. $$

So the length function we are interested in is given by

$$ p = x^2\xi^2 + x^2h(y,\eta) + x^3\tilde{h}(x, y, \eta), \qquad \tilde{h} \in C^\infty(X). $$

In coordinates (x, y, λ, μ), in ${}^0T^*X$, p is given by

$$ p = \lambda^2 + h(y,\mu) + x\tilde{h}(x, y, \mu) \tag{2.12}$$

Proposition 2.2. *Let $p \in C^\infty({}^0T^*X)$ be given by* (2.12) *and let ${}^0H_p$ be the 0-Hamiltonian vector field of p defined in* (2.10). *Let $\tilde{p}$ be the lift of p to $T^*(X \times_0 X)$. For $s > 0$, let χ_s: ${}^0T^*X \to {}^0T^*X$ be the map defined by*

$$ \chi_s(q) = \exp(s{}^0H_p)(q). $$

Then the graph of χ_s defines a smooth extendible Lagrangian submanifold of $T^*(X \times_0 X)$. Moreover the intersection

$$ \Lambda_{\mathcal{F},s} = \Lambda_s \cap T^*_{\mathcal{F}}(X \times_0 X) $$

is a smooth Lagrangian submanifold of $T^*\mathcal{F}$ which is given by $\exp(sH_{p_0})$ $(T^*_{D_0\cap\mathcal{F}}\mathcal{F})$, *where $p_0 = \tilde{p}|_{\mathcal{F}}$.*

Proof. Since, as observed above, the projection of ${}^0H_p$ to the base space vanishes at ∂X, the first part of the statement follows directly from Proposition 2.1. Thus we only need to check the part concerning $\Lambda_{\mathcal{F},s}$.

A useful way of describing the graph of χ_s is to view p as a function in $C^\infty({}^0T^*X \times {}^0T^*X)$ depending only on the variables of the second copy of ${}^0T^*X$. As pointed out above we work with the 1-form on ${}^0T^*X \times {}^0T^*X$ given in local coordinates by

$$\Sigma = \frac{\lambda}{x}\, dx + \frac{\mu}{x} \cdot dy - \frac{\lambda'}{x}\, dx' - \frac{\mu}{x'} \cdot dy'.$$

Then we can define the 0-Hamiltonian vector field of p that we also denote ${}^0H_p$ with respect to Σ as above. Then χ_s is the flow-out of the diagonal of ${}^0T^*X \times {}^0T^*X$ by ${}^0H_p$. In these local coordinates

$$p = \lambda'^2 + h(y', \mu') + x'\tilde{h}(x', y', \mu').$$

We consider projective coordinates, (x, y, t, z), where

$$t = x'/x, \qquad z = (y' - y)/x.$$

That is, the blow-down map β is

$$(x, t, y, z) \to (x, x', y, y') = (x, tx, y, y + xz).$$

Notice that in these coordinates $\mathscr{F} = \{x = 0\}$ and $D_0 = \{t = 1, z = 0\}$. Since

$$dx' = x\, dt + t\, dx, \qquad dy' = dy + x\, dz + z\, dx,$$

our coordinate transformation β on phase space is,

(2.13)

$$(x, \lambda, y, \mu, x', y', \lambda', \mu') \mapsto (x, \sigma, t, \tau, y, \tilde{\eta}, z, \zeta), \qquad \text{where}$$

$$(x, \sigma, t, \tau, y, \tilde{\eta}, z, \zeta) = \left(x, \frac{\lambda}{x} - \frac{\lambda'}{x} - \frac{\mu'}{x'} \cdot \frac{y - y'}{x}, \frac{x'}{x}, -\lambda' \frac{x}{x'}, y, \frac{\mu}{x} - \frac{\mu'}{x'}, \frac{y - y'}{x}, -\frac{x\mu'}{x'}\right).$$

Hence p lifts to

$$\tilde{p} = t^2\tau^2 + t^2 h(y + xz, \zeta) + t^3 x\tilde{h}(tx, y + xz, \zeta).$$

Therefore, if $\tilde{\omega}$ denotes the lift of the symplectic form (2.3) to $T^*(X \times_0 X)$, ${}^0H_p$ lifts to $H_{\tilde{p}}$ which is given by $\tilde{\omega}(\,\bullet, H_{\tilde{p}}) = d\tilde{p}$. In these coordinates we have

$$
\begin{aligned}
H_{\tilde{p}} = & \frac{\partial \tilde{p}}{\partial \sigma}\frac{\partial}{\partial x} - \frac{\partial \tilde{p}}{\partial x}\frac{\partial}{\partial \sigma} + \frac{\partial \tilde{p}}{\partial \tau}\frac{\partial}{\partial t} - \frac{\partial \tilde{p}}{\partial t}\frac{\partial}{\partial \tau} \\
& + \sum_j \left(\frac{\partial \tilde{p}}{\partial \tilde{\eta}_j}\frac{\partial}{\partial y_j} - \frac{\partial \tilde{p}}{\partial y_j}\frac{\partial}{\partial \tilde{\eta}_j} \right) \\
& + \sum_j \left(\frac{\partial \tilde{p}}{\partial \zeta_j}\frac{\partial}{\partial z_j} - \frac{\partial \tilde{p}}{\partial z_j}\frac{\partial}{\partial \zeta_j} \right) \\
= & \, 2t^2\tau \frac{\partial}{\partial t} - 2t(\tau^2 + h(y,\zeta))\frac{\partial}{\partial \tau} - t^2 \left(\sum_j z_j \frac{\partial \tilde{p}}{\partial y_j}(y,\zeta) \right) \frac{\partial}{\partial \sigma} \\
& + t^2 \sum_j \frac{\partial h}{\partial \zeta_j}(y,\zeta)\frac{\partial}{\partial z_j} - t^2 \sum_j \frac{\partial h}{\partial y_j}\frac{\partial}{\partial \tilde{\eta}_j} + O(x).
\end{aligned}
$$

This shows that $H_{\tilde{p}}$ is smooth and tangent to $\mathscr{F} = \{x = 0\}$.

On the other hand (2.13) gives that the diagonal $\{x = x', \, y = y', \, z = z', \, \lambda = \lambda', \, \mu = \mu'\}$ lifts to

$$\tilde{D}_0 = \{t = 1, \, z = 0, \, \sigma = 0, \, \tilde{\eta} = 0\}.$$

Clearly $\tilde{D}_0$ intersects $T^*_{\mathscr{F}}(X \times_0 X)$ transversally at

$$\tilde{D}_0 \pitchfork T^*_{\mathscr{F}}(X \times_0 X) = \{x = 0, \, t = 1, \, z = 0, \, \sigma = 0, \, \tilde{\eta} = 0\} = T^*_{D_0 \cap \mathscr{F}} \mathscr{F}.$$

Finally we observe that the projection of $H_{\tilde{p}}$ to $T^*\mathscr{F}$ is given by

$$2t^2\tau \frac{\partial}{\partial t} - 2t(\tau^2 + h(y,\zeta))\frac{\partial}{\partial \tau} + 2t^2 \sum_{ij} h^{ij}(y)\,\zeta_i \frac{\partial}{\partial \zeta_j} - t^2 \sum_j \frac{\partial h}{\partial y_j}\frac{\partial}{\partial \tilde{\eta}_j}$$

which is the Hamiltonian vector field of $t^2(\tau^2 + h(y,\zeta))$.

Since the bottom and the top faces are given respectively by $t = 0$ and $t = \infty$, we see that Λ_s does not intersect either one for $s \in \mathbb{R}$.

This concludes the proof of the proposition. ∎

Next, we rephrase Proposition 2.2 in terms of the normal operator. We have

Corollary 2.1. *Let $p \in C^\infty({}^0T^*X)$ be given by* (2.12). *Let ${}^0H_p$ denote the* 0*-Hamiltonian vector field of p. Then, for $s > 0$, the twisted graph of $\chi_s =$* $\exp(s\,{}^0H_p)$ *defines a Lagrangian submanifold $\Lambda_s \subset T^*(X \times_0 X)$ which intersects $\partial T^*(X \times_0 X)$ only over $T^*_{\mathscr{F}}(X \times_0 X)$ and does so transversally. Moreover the intersection*

$$\Lambda_s \pitchfork T^*_{\mathscr{F}}(X \times_0 X) = \Lambda_{\mathscr{F},s}$$

is a smooth Lagrangian submanifold of $T^*\mathcal{F}$ *which is given by* $\exp(sH_{p_0})(T^*_{D_0 \cap \mathcal{F}}\mathcal{F})$ *where* p_0 *is the symbol of the normal operator* Δ_0 *of* Δ, *and the symplectic structure on* $\mathcal{F}$ *is the one induced from* $T^*(X \times_0 X)$.

The kernel of the wave group is a distribution in $\mathbb{R} \times X \times X$, in particular it is not associated with the graph of a canonical transformation of ${}^0T^*X$. So we need to enlarge our class of Lagrangians to include it.

We consider the bundle $T^*\mathbb{R} \times {}^0T^*X \times {}^0T^*X$ with the one canonical 1-form

$$\alpha = \tau\, dt + \frac{\lambda}{x}\, dx + \frac{\mu}{x} \cdot dy - \frac{\lambda'}{x'}\, dx' - \frac{\mu'}{x'} \cdot dy' \tag{2.14}$$

Following [7] and [8], let $C \subset T^*\mathbb{R} \times {}^0T^*X \times {}^0T^*X$, with the form (2.14), be defined as

$$\begin{aligned} C = \{(t, \tau, x, y, \lambda, \mu, x', y', \lambda', \mu') \colon \tau + \sqrt{p(x, y, \lambda, \mu)} = 0;\\ (x', y', \lambda', \mu') = \chi_t(x, y, \lambda, \mu)\}, \end{aligned} \tag{2.14}$$

where p is defined by (2.12) and χ_t is the canonical transformation defined in Proposition 2.2. Following the proof of Propositions 2.1 and 2.2 we can prove

PROPOSITION 2.3. *The relation* C *defines a Lagrangian submanifold* Λ_C *of* $T^*\mathbb{R} \times T^*(X \times_0 X)$ *given by*

$$\Lambda_C = \{(t, \tau, Y, H) \colon \tau + \sqrt{\tilde{p}(Y, H)} = 0, (Y, H) \in \Lambda_t\}, \tag{2.16}$$

where $\tilde{p}$ *denotes the lift of* p *defined in the first copy of* ${}^0T^*X$ *and* Λ_t *is defined in Proposition 2.2. Moreover,* Λ_C *intersects the boundary only over* $\mathcal{F}$ *and*

$$\Lambda_C^0 = \Lambda_C \pitchfork (T^*\mathbb{R} \times T^*_{\mathcal{F}}(X \times_0 X))$$

is a Lagrangian submanifold of $T^*\mathbb{R} \times T^*\mathcal{F}$ *given by*

$$\Lambda_C^0 = \{(t, \tau, Y_0, H_0) \colon \tau + \sqrt{p_0(Y_0, H_0)} = 0, (Y_0, H_0) \in \Lambda_{\mathcal{F}, t}\}, \tag{2.17}$$

where, as in Proposition 2.2, p_0 *is the restriction of* $\tilde{p}$ *to* $\mathcal{F}$.

3. FOURIER INTEGRAL OPERATORS

A Lagrangian distribution with respect to an extendible Lagrangian $\Lambda \subset T^*(X \times_0 X)$, or $\Lambda \subset T^*\mathbb{R} \times T^*(X \times_0 X)$, is defined to be the restriction to $X \times_0 X$ of a distribution which is Lagrangian with respect to an extension Λ_e of Λ across $\mathscr{F}$ to $X \times_0 X$. For simplicity, as it makes no difference to the applications here, we assume that the unrestricted distribution is supported in the interior of the double of the manifold $X \times_0 X$ across $\mathscr{F}$; i.e., its support does not intersect the top or bottom faces.

Our class of distributions now has a pair of natural symbols. The first is the ordinary symbol of a Lagrangian distribution in the interior, which will be a smooth section of the Maslov bundle tensored with the half-density bundle over Λ, which is smooth up to the boundary of Λ. It follows from the transversality assumption that the restriction of a Lagrangian distribution to the front face is in fact Lagrangian with respect to Λ_0. The symbol of this restriction will give the second natural symbol. Finally, we want to define a filtration which corresponds to the order of vanishing at the front face. Let R be a boundary defining function for the front face in $X \times_0 X$. We define $I^{m,s}(\Lambda)$ to be equal to $R^s I^m(\Lambda)$. The symbol at the front face, $\sigma_s^f(u)$, is then defined to be the restriction of $R^{-s}u$ to the front face. This is of course dependent on the choice of R but is invariant as a section of the normal bundle raised to the power s. In what follows we will fix a product decomposition in which 1.1 holds. This will give a defining function R of the front face, so we will ignore this coordinate dependence. The class of ordinary Lagrangian symbols will just be a pair of elements $\sigma_{m,s}(u) = (R^s\sigma_m(u), \sigma_s^f(u))$ of the usual symbol class with the restriction that $\sigma_m(u)$ restricted to the front face equals $\sigma_s^f(u)$. (See Fig. 2)

We can see the independence of the class from the choice of extension of the Lagrangian submanifold as follows. We work in the double space and suppose Λ_1, Λ_2 both extend Λ. Suppose $u \in I^{m,s}(\Lambda_1)$; then we can take $v \in I^{m,s}(\Lambda_2)$ with the same symbol on Λ. The difference is then of order $m-1$ on Λ. Now, by the transverality assumption, each of $R^{-s}u$, $R^{-s}v$ will

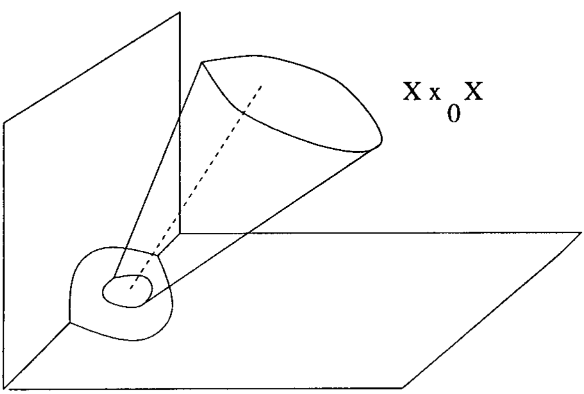

FIG. 2. The support of the wave kernel at time t.

have well-defined symbols at the front face and these symbols coincide. We can iteratively remove the principal symbols to get w such that $R^{-s}(u-w)$ is of order $-\infty$ everywhere on Λ and on the front face. We can therefore now iterate again removing only functions smooth on the double space to get $u-w$ smooth and rapidly decaying at the front face. The result is now clear.

DEFINITION 3.1. If $\chi\colon {}^0T^*X \to {}^0T^*X$ is a liftable 0-canonical transformation, and Λ_χ is the Lagrangian submanifold of $T^*(X\times_0 X)$ given by the graph of χ, as in Proposition 2.1, we define the space of 0-Fourier integral operators associated to χ as consisting of the kernels

$$I_0^{m,s}(X,\chi,{}^0\Omega^{1/2}) = \{K \in I^{m,s}(X\times_0 X, \Lambda_\chi, {}^0\Omega^{1/2});$$
$$K \text{ vanishes in a neighbourhood of } \partial(X\times_0 X)\backslash\mathscr{F}\}.$$

If C is given by (2.15) and Λ_C is the Lagrangian defined in Proposition 2.3, then we define

$$I_0^{m,s}(\mathbb{R}\times X, X; C, {}^0\Omega^{1/2}) = \{K \in I^{m,s}(\mathbb{R}\times X\times_0 X, \Lambda_C, {}^0\Omega^{1/2});$$
$$K \text{ vanishes in a neighbourhood of}$$
$$\partial(\mathbb{R}\times X\times_0 X)\backslash\mathbb{R}\times\mathscr{F}\}.$$

Since the Lagrangians Λ_χ and Λ_C intersect the corresponding fibers over the front face transversally, we can define the normal operators of elements $F\in I_0^{m,s}(X,\chi,{}^0\Omega^{1/2})$, $F\in I_0^{m,s}(\mathbb{R}\times X, X, C, {}^0\Omega^{1/2})$ as in (1.4). Moreover we find that $N_p(F)$ is a Lagrangian distribution with respect to $\Lambda_{\chi_0} = \Lambda_\chi \cap T^*_{\mathscr{F}}(X\times_0 X)$ or $\Lambda_C^0 = \Lambda_C \cap T^*\mathbb{R}\times T^*_{\mathscr{F}}(X\times_0 X)$.

We want to understand the mapping properties of these operators under the action of zero differential operators—particularly the Laplacian and the wave operator. A line by line inspection of the proof of Proposition 5.19 of [21] gives its analogue for 0-Fourier integral operators. We have

PROPOSITION 3.1. *The normal operator* (1.4) *defines an exact sequence*

(3.1)
$$0 \to I_0^{m,1}(\mathbb{R}\times X, X; C, {}^0\Omega^{1/2}) \to I_0^{m,0}(\mathbb{R}\times X, X; C, {}^0\Omega^{1/2}) \to I^m(\mathscr{F}, \Lambda_C^0, \Omega^{1/2})$$

such that for any differential operator $P\in \operatorname{Diff}_0^m(X)$ *and any* $F\in I_0^{m,0}(\mathbb{R}\times X, X; C, {}^0\Omega^{1/2})$

$$N_p((D_t^2 - P).F) = (D_t^2 - N_p(P))\cdot N_p(F). \tag{3.2}$$

Now we come to the main result of this section.

THEOREM 3.1. *For $t \in \mathbb{R}$, let C be the relation defined by* (2.15). *The wave group $U(t)$ satisfies*

$$U(t) = \cos\left(t\sqrt{\Delta - \frac{n^2}{4}}\right) \in I_0^{-1/4,\,0}(\mathbb{R} \times X, X; C, {}^0\Omega^{1/2}). \tag{3.3}$$

Proof. This is very similar in nature to the construction in Section 7 of [21]. Using Proposition 3.1 and Eq. (1.6) we find that the normal operator $U_0(p, t) = N_p(U(t))$ satisfies

$$\begin{gathered} \left(D_t^2 - \Delta_0 - \frac{n^2}{4}\right) U_0(p, t) = 0, \\ U_0(p, 0) = \delta(0_p), \qquad D_t U_0(p, 0) = 0, \end{gathered} \tag{3.4}$$

where 0_p is the center of F_p and Δ_0 is the normal operator of Δ. Since 0_p is away from the boundaries of $\mathscr{F}$ and Λ_C^0 does not intersect the boundaries for finite t, it follows from the usual theory of Fourier integral operators that $U_0(t) \in I^{-1/4}(\mathbb{R} \times \mathscr{F}, \Lambda_C^0, \Omega^{1/2})$; see for example [7].

By the surjectivity of the map (3.1), we can pick an element, u_0, of $I_0^{-1/4,\,0}(\mathbb{R} \times X, X; C, {}^0\Omega^{1/2})$, with $N_p(u_0) = U_0(t)$, so that

$$v_0 = U(t) - u_0 \in I_0^{-1/4,\,1}(\mathbb{R} \times X, X; C, {}^0\Omega^{1/2}).$$

Let x' be, as above, a defining function of the second copy of X. Now $(x')^{-1} v_0 \in I^{-1/4,\,0}$, and, as v_0 is supported away from the bottom face, no difficulties are introduced. We now solve on the front face to get $w_1 \in I_0^{-1/4,\,0}$ satisfying

$$\begin{gathered} \left(D_t^2 - \Delta_0 - \frac{n^2}{4}\right) N_p(w_1) = N_p((x')^{-1} v_0), \\ N_p(w_1)(0) = N_p((x')^{-1} v_0)), \qquad D_t N_p(w_1)(0) = D_t(N_p((x')^{-1} v_0))(0). \end{gathered} \tag{3.5}$$

We let $u_1 = x' w_1$. We then have, by the uniqueness of the solution to (3.5), that

$$U(t) - u_0 - u_1 \in I_0^{-1/4,\,2}(\mathbb{R} \times X, X; C, {}^0\Omega^{1/2})$$

We can now iterate at each level by considering $(x')^{-k}$ times the error achieved. Since u_j is supported away from the top and bottom faces,

$\tilde{u}_j = \frac{x'}{x} u_j$ is a Fourier integral operator in the same class. Thus we have found

$$U(t) - \sum_{j=0}^{k} x^j \tilde{u}_j \in I_0^{-1/4, k+1}(\mathbb{R} \times X, X; C, {}^0\Omega^{1/2}). \tag{3.6}$$

Asymptotically summing, we achieve an error in $I_0^{-1/4, \infty}(\mathbb{R} \times X, X; C, {}^0\Omega^{1/2})$ and an error in the Cauchy data which vanishes to infinite order at the front face, and which is pseudo-differential operator of order zero.

We can now extend this error term to be identically zero across the front face and remove it in the usual way using Hörmander's Lagrangian calculus. See for example Theorem 1.1 of [7]. ∎

4. ASYMPTOTICS OF THE WAVE TRACE

Once the wave group is known to be a 0-Fourier integral operator, it is natural to ask whether the techniques of [4, 7, 14] developed to analyze the trace of the wave group on compact manifolds without boundary can be extended to asymptotically hyperbolic manifolds. This first difficulty is that the operator $U(t)$ is not trace class, so we need to define a natural regularization which has a trace. As above, we use $U(t, w, w')$ to denote the kernel of $\cos(t\sqrt{\Delta - n^2/4})$. We begin by observing that for $w, w' \notin \partial X$, the restriction of $U(t, w, w')$ to the diagonal is well defined. Indeed following the argument in Section 1 of [7], we let

$$i_\Delta : \mathbb{R} \times \Delta \to \mathbb{R} \times X \times X$$

$$(t, w) \mapsto (t, w, w).$$

The pull-back i_Δ^* is a Fourier integral operator of order $\frac{1}{4}n$ defined by the canonical relation

$$WF'(i_\Delta^*) = \{(((t, \tau), (w, \zeta + \eta)), ((t, \tau), (w, \zeta), (w, \eta)))\}.$$

If $w, w' \notin \partial X$, because $\tau \neq 0$ when $((t, \tau), (w, \zeta), (w, \eta)) \in WF(U)$, we can apply Hörmander's transversal composition theorem; see for example Theorem 2.5.11' of [13] to conclude that $i_\Delta^* U(t)$ is a well defined distribution in $\mathbb{R} \times (X \backslash \partial X)$ and

$$WF(i_\Delta^* U(t)) \subset \{((t, \tau), (w, \zeta - \eta)) : \tau + q(w, \zeta) = 0, (w, \zeta) = \chi_t(w, \eta)\}.$$

For $\varepsilon > 0$, let $X_\varepsilon = \{x > \varepsilon\}$, where as above, x is a defining function of ∂X. Let

$$\pi\colon \mathbb{R} \times X_\varepsilon \to \mathbb{R}$$
$$(t, w) \mapsto t;$$

then integration over w is equal to the push-forward π_*, so it is a Fourier integral operator defined by the canonical relation

$$WF'(\pi_*) = \{((t, \tau), ((t, \tau), (w, 0)))\}.$$

Applying Hörmander's theorem again we conclude that

$$T_\varepsilon(t) = \int_{x > \varepsilon} U(t, w, w) = \pi_*(i_\Delta^* U(t))$$

is a well-defined distribution and

$$WF(T_\lambda) = \{(t, \tau)\colon \tau \tilde{\Lambda} 0 \text{ and } (w, \zeta) = \chi_t(w, \zeta) \text{ for some } (w, \zeta) \text{ with } x(w) > \varepsilon\}.$$

Notice that the restriction of the half density factor in $U(t, w, w')$ to the diagonal gives a 1-density in X. In particular we have proven

THEOREM 4.1. *For $\varepsilon > 0$, the singular support of T_ε is contained in the set of periods of closed geodesics in X_ε.*

PROPOSITION 4.1. *There exists $\varepsilon_0 > 0$ such that all closed geodesics of (X, g), with period greater than zero, are contained in X_{ε_0}.*

Proof. We will show that for ε small, a geodesic γ which intersects $\{x < \varepsilon\}$ cannot be closed. We know from Proposition 2.1 of [16] that for ε small, there exists a product decomposition $X \sim \partial X \times [0, \varepsilon)$ in which 1.1 holds. In these coordinates, the geodesic flow is generated by the Hamiltonian function

$$\sigma = x^2 \xi^2 + x^2 \sum_{i, j=1}^{n-1} h^{ij}(x, y)\, \eta_i \eta_j.$$

It is convenient to rescale the coordinates (ξ, η) by $\lambda = x\xi$, $\mu = x\eta$ and leave (x, y) unchanged. Then the canonical 1-form $\alpha = \xi dx + \eta \cdot dy$ and the Hamiltonian are respectively rescaled to

$$^0\alpha = \lambda \frac{dx}{x} + \mu \frac{dy}{x}$$

and

$$\bar{\sigma} = \lambda^2 + \sum_{i,\, j=1}^{n-1} h^{ij}(x, y)\, \mu_i \mu_j.$$

The 0-Hamiltonian vector field H of $\bar{\sigma}$ is given by

$$d^0\alpha(\,\bullet, H) = d\bar{\sigma}.$$

We find that, see [16],

$$H = 2\lambda \left(x \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial \lambda} \right) - \left(2h^{-1} + x \frac{\partial}{\partial x} h^{-1} \right) \frac{\partial}{\partial \lambda}$$
$$+ x \sum_i \left(\frac{\partial}{\partial \mu_i} h^{-1} \frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_i} h^{-1} \frac{\partial}{\partial \mu_i} \right),$$

where $|\mu|^2 = h^{-1}(x, y, \mu) = \sum_{i,\, j=1}^{n-1} h^{ij}(x, y)\, \mu_i \mu_j$. Now we restrict this to the co-sphere bundle

$$\lambda^2 + |\mu|^2 = 1,$$

which is invariant under the flow of H.

In particular, if s is the arc-length parameter on $\tilde{\gamma}$, the coordinates x and λ on $\tilde{\gamma}$ satisfy

$$\frac{d}{ds} x(s) = 2\lambda(s)\, x(s),$$

$$\frac{d}{ds} \lambda(s) = 2\lambda^2 - 2h^{-1} - x \frac{\partial}{\partial x} h^{-1}.$$

If the curve γ is closed and intersects $\{x \in \varepsilon\}$, there exists $\delta \in (0, \varepsilon)$ such that γ intersects $S_\delta = \{x = \delta\}$ at two distinct points. Therefore there exists s_0, with $x(s_0) > 0$, where $x(s)$ has a minimum. The first equation says that $\frac{d}{ds} x(s_0)$ is equal to zero at an interior point if and only if $\lambda(s_0) = 0$. But in that case, $h^{-1} = 1$ and the second equation gives that for small ε, $\frac{d}{ds} \lambda(s_0) < 0$. Therefore $\frac{d}{ds} \lambda(s) < 0$ for s close to s_0. Hence $\lambda(s)$ will be negative for $s > s_0$ close to s_0. But then $\frac{d}{ds} x(s) < 0$ for $s > s_0$ close to s_0. Therefore $x(s_0)$ is not a minimum and hence γ cannot be closed. ∎

As a consequence of Theorem 4.1 and Proposition 4.1 we have

Corollary 4.1. *There exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, the singular support of T_ε is contained in the set of periods of closed geodesics of X.*

Let $\tilde{u}_j \in I_0^{-1/4, j}(\mathbb{R} \times X, X; C, {}^0\Omega^{1/2})$ be the operators defined (3.6). The argument used above can be applied to show that

$$I_j(t, \lambda) = \int_{x > \varepsilon} x^j \tilde{u}_j(t, w, w)$$

is well defined and its singular support is contained in the set of periods of closed geodesics in X_ε.

Since Λ_C meets $\mathscr{F}$ transversally, we observe that the only obstacle for the convergence of $I_j(t, \varepsilon)$, as $\varepsilon \to 0$, is the density factor in u_j, which behaves as x^{-n} when $x \to 0$. Thus, for $j \geqslant n$ the integral

$$\int_{x > 0} x^j \tilde{u}_j(t, w, w)$$

converges. By taking the Taylor's expansion of $\tilde{u}_j(t, w, w)$ as $x \to 0$, it follows that there exist constants C_j, $j = 1, ..., n-1$, such that the limit.

$$0 - \operatorname{tr}(U(t)) = \lim_{\lambda \to 0} \left[\left[\int_{x > \varepsilon} U(t, w, w) - \sum_{j=1}^{n-1} C_j \varepsilon^{-j} + C_0 \log \varepsilon \right] \right],$$

exists. This is called the zero-trace of $U(t)$, in analogy with the b-integral of [26]; see also the notion of b-trace of [5]. It is a Hadamard regularization and it obviously depends on the choice of the boundary defining function, x, but it gives a natural regularization of the trace of $U(t)$. In the case of Riemann surfaces, the notion of zero-trace was introduced and studied in depth by Guillopé and Zworski [10, 11]. In particular a much more precise version of Theorem 4.2 below was established in [11].

The following is another consequence of Theorem 4.1 and Proposition 4.1.

THEOREM 4.2. *The singular support of* $0 - \operatorname{tr}(U(t))$ *is contained in the set of periods of closed geodesics of* (X, g).

Next we analyze the behaviour of $0 - \operatorname{tr}(U(t))$ as $t \to 0$. Suppose that the set of periods of closed geodesics is contained in (t_0, ∞). Let $\rho \in C_0^\infty(\mathbb{R})$ be such that $\rho(t) = 1$ for $|t| > t_0/2$ and $\rho(t) = 0$ for $|t| > 2t_0/3$.

Since the arguments in [14], see also the proof of Proposition 2.1 of [7], are entirely local, we can apply them directly to prove

PROPOSITION 4.2. *There exist* $w_k \in \mathbb{R}$, $k = 0, 1, ...,$ *with* $\omega_0 = \operatorname{vol}(X_\lambda)$, *such that*

$$\int_{\mathbb{R}} e^{t\mu} \rho(t)\, T_\varepsilon(t)\, dt \sim \frac{1}{(2\pi)^{n-1}} \sum_{k=0}^{\infty} \omega_k\, \mu^{n-2k-1}, \tag{4.1}$$

for $\mu \to \infty$ *and is rapidly decreasing if* $\mu \to -\infty$.

The analogous result for $0-\mathrm{tr}(U(t))$ follows directly from its definition and 4.1. We obtain

PROPOSITION 4.3. *There exist $\theta_k \in \mathbb{R}$, $k=0, 1, ...,$ such that*

$$\int_{\mathbb{R}} e^{t\mu}\rho(t)\, 0-\mathrm{tr}(U(t))\, dt \sim \frac{1}{(2\pi)^{n-1}} \sum_{k=0}^{\infty} \theta_k\, \mu^{n-2k-1}, \tag{4.2}$$

for $\mu \to \infty$ and is rapidly decreasing if $\mu \to -\infty$.

Observe that

$$\theta_0 = \lim_{\lambda \to 0} \left(\int_{x>\lambda} d\,\mathrm{vol}_g - \sum_{j=1}^{n-1} d_j \varepsilon^{-j} - d_0 \log \lambda \right)$$

where d_j, $j=0, 1, 2, ..., n-1$ are the unique real numbers such that the limit exists. This is called the zero-volume of X and is denoted $0-\mathrm{vol}(X)$.

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